Math 211    Problem Set 2

1. Prove: If \((A, \leq)\) is a partially ordered set then \(A\) has at most one minimum element.

2. Prove: If \((A, \leq)\) is a partially ordered set and \(K \subseteq A\) the \(K\) has at most one least upper bound.

3. Assume that \((A, \leq)\) is a partially ordered set, that \(a \in A\) and that \(K \subseteq A\). Give the negations of the following sentences in a positive form. Negate the symbolic form of the sentence that is given.
   
   (a) “\(a\) is an upper bound of \(K\).” Symbolic form: “\(\forall x \in K, x \leq a\).”
   
   (b) “\(K\) has an upper bound.” Symbolic form: “\(\exists y \in A\) such that \(\forall x \in K, x \leq y\).”
   
   (c) “\(K\) has a least upper bound.” Symbolic form: “\(\exists z \in A\) such that \(z\) is an upper bound for \(K\) and \(\forall y \in A\), if \(y\) is an upper bound for \(A\), then \(z \leq y\).”

   In part 3c you may use the words “least upper bound” without translating into symbolic form.

4. Prove or find a counter example: If \((A, \leq)\) is a partially ordered set and \(K \subseteq A\) then \(K\) has at most one upper bound.

5. Prove or find a counter example: If \((A, \leq)\) is a partially ordered set and \(K \subseteq A\) and \(a\) is an upper bound of \(K\) which is an element of \(K\), the \(a\) is a least upper bound for \(K\).

6. Define the ordering \(\leq\) on \(\mathbb{N}\) by \(a \leq b\) if and only if \(b = 2^k a\) for some integer \(k \geq 0\).
   
   (a) Prove that \(\leq\) is a partial ordering on \(\mathbb{N}\).
   
   (b) Give an example of a subset of \(\mathbb{N}\) with more than two elements which has an upper bound.
   
   (c) Give a subset of \(\mathbb{N}\) with no lower bound.

7. Define the relation \(\sqsubseteq\) on \(\mathbb{R} \times \mathbb{R}\) by \((x, y) \sqsubseteq (z, w)\) if and only if \(x \leq z\) and \(y \leq w\). (In this definition \(\leq\) represents the usual ordering on \(\mathbb{R}\).)
   
   (a) Prove that \(\sqsubseteq\) is a partial ordering on \(\mathbb{R}\).
   
   (b) Does the set \(\{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}\) have an upper bound?
   
   (c) Does the set \(\{(x, y) : y = 0\}\) have an upper bound?
   
   (d) Does the set \(\{(x, y) : x = 1 \text{ and } 1 \leq y \leq 2\}\) have an upper bound?

8. Which of the following are linear orderings?
   
   (a) The ordering \(\leq\) defined on \(\mathbb{N}\) by \(m \leq n\) if and only if \(m < 2n\)
   
   (b) The ordering \(\sqsubseteq\) on \(\mathbb{N} \times \mathbb{N}\) defined by \((a, b) \sqsubseteq (c, d)\) if and only if \(a \leq c \text{ or } (a = c \text{ and } b \leq d)\). (In part 8b only, \(\leq\) denotes the usual ordering on \(\mathbb{N}\).)