Applications of Derivatives

I. Curve Sketching and How to Do It

- Find the intercepts
- Find the first derivative and do a sign chart for it
- Find the second derivative and do a sign chart for it
- Combine the two sign charts to get what each piece looks like
- If $x = a$ is an $x$-value for which the function does not exist, find $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.
- Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.
- Using the information above, sketch the graph

II. Optimization

1. Read the question!
2. If possible, draw and label a diagram
3. Using the information in the problem, come up with a function that we are trying to optimize (the thing that’s the biggest, smallest, fastest, shortest, largest, etc.). This could involve
   - Using the Pythagorean theorem
   - Using trig functions and triangles
   - Using similar triangles
   - Making a table to see a pattern
4. Maximums and minimums of functions can only be found in three places:
   - Places where $f'(x) = 0$
   - Places where $f'(x)$ is not defined
   - Endpoints (what’s the biggest $x$ can be? what’s the smallest $x$ can be?)
   So, we need to find all of these $x$-values.
5. Either by using a sign chart for the first derivative (this is called the ‘first derivative test’, or by using the second derivative test, determine whether the $x$-values you’ve found are local minimums or local maximums or neither
6. By comparing the function values, you can now just pick out the highest and lowest values
III. Related Rates:

1. If possible, draw and label a diagram/picture.
2. A rate is usually given in the problem; write that rate as a derivative.
3. A rate is always being asked for; write that rate as a derivative.
4. Find an equation linking the variables whose derivatives are described in parts 2. and 3. This should involve the picture, or a geometrical formula, or similar triangles.
5. Using implicit differentiation, take the derivative of both sides with respect to the necessary variable (this will be the variable that is in the denominators of the derivatives in 2. and 3.)
6. Plug in what you know, figure out what you have to, and solve for the requested rate.

IV. Linear Approximations: When trying to approximate a function’s value (so you’re trying to find a value for \( f(x_0) \), but it’s too difficult, for example with roots, or \( \ln \) or trig functions):

1. Figure out what the function is, and what number you’re trying to plug into it
2. Find a value, \( a \), near to \( x_0 \) that’s easy to plug into the function
3. Find the equation of the tangent line to the graph of \( f(x) \) at \( x = a \). This is called the linearization \( L \) of \( f \) at \( a \):
   \[
   L(x) = f(a) + f'(a)(x - a).
   \]
4. We know that for all \( x \)-values near \( a \), \( f(x) \approx L(x) \), and \( L(x) \) is a linear function, so, easy to plug into. So, to get our approximation, we find \( L(a) \).
V. Newton’s method: If you’re trying to solve an equation and can’t do it algebraically:

- Make sure you can’t do it algebraically!
- Rewrite your equation so that it is of the form
  \[ f(x) = 0 \]
- Make an initial guess. Call it \( x_0 \). You might use the graph to pick a decent one.
- Come up with better guesses by using the formula
  \[ x_{n+1} = -\frac{f(x_n)}{f'(x_n)} + x_n \]

Each time you do this, you’re said to have done an *iteration*. Each time you do this, your approximation to a solution should become better and better.
VI. L’Hospital’s Rule: If you’re trying to find a limit, so that (after rewriting if necessary!), you get either \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \) or \( \lim_{x \to a} \frac{f(x)}{g(x)} = \pm\infty \), you can try to evaluate the limit by

- Finding \( f'(x) \) and \( g'(x) \) (This IS NOT THE QUOTIENT RULE!)
- Try finding \( \lim_{x \to a} \frac{f'(x)}{g'(x)} \). If you can evaluate this limit, it is equal to the original limit.

Examples:

1. \( \lim_{x \to 0} \frac{\sin(2x)}{3x} \)
2. \( \lim_{x \to 3} \frac{\ln(x-1)}{(x-2)^2} \)
3. \( \lim_{x \to 2} \frac{\ln(x-1)}{(x-2)^2} \)
4. \( \lim_{x \to 0^+} x \ln(x) \)
5. \( \lim_{x \to 1} \left( \frac{1}{x-1} - \frac{1}{\ln(x)} \right) \)
6. \( \lim_{x \to \infty} \frac{e^x}{x^2} \)
7. \( \lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} \)
8. \( \lim_{x \to 0} \frac{\sin(x)}{x - \tan(x)} \)
9. \( \lim_{x \to 0^+} x^2 \ln(x) \)
10. \( \lim_{x \to 0} \left( \frac{1}{e^x - 1} - \frac{1}{x} \right) \)