Week 5 – Equations Part 2

Many images below are excerpts from the multimedia textbook. You can find them there and in your textbook in sections 2.6 and 2.7.

This week we will add several new types of equations to our list. In section 2.7 we learn how to solve inequalities.

Let us summarize how we solve quadratic equations.

**Steps of Solving a Quadratic Equation:**

1. Identify the equation as quadratic in $x$.
2. Collect all terms on the left hand side and set the equation equal to zero. Arrange in decreasing order of powers of $x$. (Collect like terms.)
3. Decide if the expression factors readily.
4. If it does, proceed to factor it into linear factors and set each factor equal to zero and solve the resulting linear equations.
5. If it does not factor easily, use the quadratic formula.

Don't underestimate step 1 in this list. Very often it is the most important step. You need to realize what kind of equation you are trying to solve. If you don’t recognize the type, you are very likely to use the wrong method.

**Solving Rational Equations**

A rational equation contains rational expressions. There is really only one rule to remember in dealing with rational equations:

**To solve a rational equation multiply both sides with the LCD!**

**Discussion Question 1:**
Work through Example 1 in section 2.6 on page 208. What is the first step in solving the equation? Is this in fact a rational equation? What is the effect of multiplying with 6?

**Discussion Question 2:**
Steps of Solving a Rational Equation:

1. Identify the equation as rational in $x$.
2. Find the Lowest Common Denominator of all rational expressions and multiply both sides of the equation with the LCD.
4. The resulting equation is not a rational equation any more. Identify the type of the equation and choose appropriate steps.
5. At the end check the values you found, to make sure that the equation is defined for them. Not all values might in fact be solutions of the original equation.

It is important to really find the LCD. If you simply multiply with the product of all occurring denominators you are left with an equation, which may be too difficult to solve. For example it may be cubic instead of quadratic and you might not see how it factors.

Solving Radical Equations

Radical equations involve radical expressions. The important rule to eliminate the radical expression is to square the whole equation if it’s a square root, to cube if it’s a cube root, etc. BUT only do that step AFTER isolating the radical expression! Also, squaring can introduce extraneous solutions. That means one needs to check all found values to make sure that they indeed satisfy the original equation.

Work through Example 3 in section 2.6 on page 210.

In this example there was just one square root and squaring after isolating the radical expression, eliminates the radical. In the next example two radical expressions are present from the beginning, and if we isolate one and square, one radical is still left and one has to repeat the sequence of steps one more time.

Work through Example 4 in section 2.6 on page 211.
Discussion Question 3:
After you have worked through Examples 3 and 4 in section 2.6, compare the solution in the textbook with the following approach:
As first step square both sides of the equation. Compare the result with the result after squaring in the example. Explain what you found.

Steps of Solving a Radical Equation with one Square Root:

1. Identify the equation as radical in $x$.
2. Isolate the radical on the left hand side.
3. Square both sides of the equation.
4. Simplify.
5. The resulting equation is not a radical equation any more. Identify the type of the equation and choose appropriate steps.
6. At the end check the values you found, to make sure that they in fact satisfy the original equation.

Steps of Solving a Radical Equation with two Square Roots:

1. Identify the equation as radical in $x$.
2. Isolate the one of the radicals on the left hand side.
3. Square both sides of the equation.
4. Simplify.
5. There should be only one radical left at this point. Now follow the instructions for solving a radical equation with one square root.

Solving Equations with Absolute Values

In order to work with absolute values you have to realize that it is an abbreviation: $|x|$ is equal to $x$ if $x$ is greater than or equal to zero and it is equal to $-x$ if $x$ is less than zero. So there are always those two cases to consider.

For $a > 0$:

$$|X| = a \text{ is equivalent to } X = -a \text{ or } X = a.$$
Solving Inequalities

In understanding to solve inequalities it might be best to look at our methods of solving equalities graphically.

For example let's solve $4x^2 - 9 = 7$

We proceed by graphing the function $y = 4x^2 - 9$ and also $y = 7$ in the same coordinate system.

The solutions of the equation are those $x$-values for which the $y$-values are equal as computed in both functions, or in other words, the $x$-values of the points of intersection of the two functions.

In this case the solutions are $x = 2$ and $x = -2$.

If we want to solve the inequality $4x^2 - 9 < 7$ then we need to find all those values of $x$ for which the $y$-value as computed on the function $4x^2 - 9$ is less than 7. That means we need to find all those $x$-values for which the graph of $y = 4x^2 - 9$ is below the graph of $y = 7$. 
We conclude that the solution set of the inequality $4x^2 - 9 < 7$ consists of those points on the $x$-axis marked blue in the above sketch. It is the interval from $-2$ to $2$, in set notation $\{x \mid -2 < x < 2\}$ and in interval notation $(-2, 2)$.

The key idea here was to first find the points of intersection of the two functions involved, and then figuring out for which points on the $x$-axis the inequality holds true. We are really breaking up the $x$-axis into three intervals, subdivided by the $x$-values of the two points of intersection. The other two intervals are: $(-\infty, -2)$ and $(2, \infty)$ on both $4x^2 - 9 > 7$.

The reason why the points of intersection make the divisions is that those are the only points at which the relative position of the functions can switch. That means where $4x^2 - 9 < 7$ can change into $4x^2 - 9 > 7$. This conclusion holds only true for functions, which are continuous. They can be sketched without lifting up the pen. If one of the functions makes jumps then the relative position of the functions to each other can change at that point as well.

To illustrate the point let’s change the functions and consider the function $f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ 5x + 3 & \text{if } x > 0 \end{cases}$ and then ask for which $x$ values is $f(x) > 2$.

The graph is given below:
We have only one point of intersection, since $x = -\sqrt{2}$ is the only value for which $f(x) = 2$, BUT the function jumps at $x = 0$ from 0 to 3. It is discontinuous. So the solution set for our problem is $(-\infty, -\sqrt{2}) \cup (0, \infty)$.

The lesson, which should be learned from the last example, is:
When you solve an inequality, always have an idea how the graphs of the different functions involved relate to each other.

It is often convenient to manipulate the inequality algebraically before one makes the graphs.
Here is an overview over the rules:

**Principles for Solving Inequalities**

For any real numbers $a$, $b$, and $c$:

**The Addition Principle for Inequalities:** If $a < b$ is true, then $a + c < b + c$ is true.

**The Multiplication Principle for Inequalities:** If $a < b$ and $c > 0$ are true, then $ac < bc$ is true. If $a < b$ and $c < 0$ are true, then $ac > bc$ is true.

Similar statements hold for $a \leq b$.

This last statement means:
If an inequality is multiplied or divided by a negative number, the inequality sign switches.

Work through the Example 1 in section 2.7 on page 215.
Both functions involved are lines, and you should be able to make a good graph by hand and follow along the algebra.

**Work through the Example 2 in section 2.7 on page 216.**

This example shows how to handle compound inequalities with an "AND". One can conveniently do it in one line, dealing with both inequalities at the same time. The "AND" means that BOTH inequalities have to hold true.

Example 3 deals with a situation where only one of the conditions needs to hold true, this is an "OR".

**Work through the Example 3 in section 2.7 on page 216.**

Inequalities involving absolute values are really compound inequalities of either form, "AND" or "OR" depending on the inequality sign:

For $a > 0$:

- $|X| < a$ is equivalent to $-a < X < a$.
- $|X| > a$ is equivalent to $X < -a$ or $X > a$.

Similar statements hold for $|X| \leq a$ and $|X| \geq a$.

**Work through the Example 4 in section 2.7 on page 217.**

If all these rules seem confusing:

Just forget the rules and look at the graphs as I tried to explain in the first couple of examples. The rules in the textbook, just try to give you a rule for each situation. But sometimes it is better to understand the overall principle and use it in all cases.

**Watch the Video for section 2.7.**