Week 3 – Functions – Transformations

We are starting with the "proper College Algebra" material this week and we are jumping right into the heart of the course. This week contains the central ideas of the course. Much of the remainder of the time will be spend to really understand these ideas and to apply them to a variety of examples.

The idea of a function is the central concept of the whole semester. This week we will concentrate on generating new functions from old ones. That means we will start out with a list of basic functions and then we will learn techniques how to build up more complicated functions from these few standard examples.

Many images below are excerpts from the multimedia textbook. You can find everything there and in your textbook in sections 1.4, 1.5 and 2.1.

List of Basic Functions

1. The simplest basic function is the constant function \( y = 0 \). Its graph is a horizontal line, in fact the x-axis.
2. The linear function \( y = x \) is also quite familiar to us. Its graph is the 45-degree line.
3. The quadratic function \( y = x^2 \) is surely also familiar to us. Its graph is called the normal parabola, opening up, with vertex at the origin.

4. The square root function \( y = \sqrt{x} \) has a graph related to the quadratic function. The right branch of the normal parabola is just turned on its side. The vertex is also at the origin. This function is only defined for non-negative numbers, since the square root of a negative number is not a real number. So there simply are no function values to the left of the y-axis.
5. The **cubic function** $y = x^3$ has a quite different shape. It resembles a stretched S shape flipped and turned on its side. The turning point of the S is at the origin.

6. The **reciprocal function** $y = \frac{1}{x}$ has the most complicated shape so far. First it is not defined when $x = 0$, because division by zero is not allowed. Since the result becomes large when one divides by a small number the function values approach positive or negative infinity ($\pm\infty$) as the x-value gets close to zero. As the x-values get large (positive or negative) the y-values approach zero, since the reciprocal of any large number is close to zero. (To describe the shape we will say that the x-axis and the y-axis are asymptotes of the function.)
7. The **absolute value function** $y = |x|$ is the last of the basic functions so far. We encountered the shape already in week 1. The graph looks like a V, opening up, with the vertex at the origin.

You need to be able to draw these 7 basic functions by hand, describe the shapes as above with words and recognize the shapes, when you see their graphs.

**The Algebra of Functions**

We can combine two functions with addition, subtraction, multiplication and division to obtain new functions. Notice that both functions have to be defined to be able to combine them. The domain of a combination is the intersection of the domains of the two original functions.
**Sums, Differences, Products, and Quotients of Functions**

If \( f \) and \( g \) are functions and \( x \) is in the domain of each function, then

\[
(f + g)(x) = f(x) + g(x), \quad (f - g)(x) = f(x) - g(x), \\
(fg)(x) = f(x) \cdot g(x), \quad (f/g)(x) = f(x)/g(x), \\
\text{provided } g(x) \neq 0.
\]

It is useful to view the concept of the sum of two functions graphically. In the graph below, we see the graphs of two functions \( f \) and \( g \) and their sum, \( f + g \). Consider finding \((f + g)(4) = f(4) + g(4)\). We can locate \( g(4) \) on the graph of \( g \) and use a compass to measure it. Then we move that length on top of \( f(4) \) and add. The sum gives us \((f + g)(4)\).

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Work through Example 8 in section 1.4 on page 114.

Discussion Question 1:
Describe the difference of finding the sum of two functions algebraically and graphically. How would you find the sum of two functions if you had tables for both of them?

Example. The graphs below show the functions \( y = f(x) \) in blue, \( y = g(x) \) in red and \( y = f(x)+g(x) \) in purple.
For your information: \( f(x) = \sin(10x) \)  \( g(x) = x^2 \).

**Discussion Question 2:**
Explain how you can obtain the purple graph by hand when you are given the blue and the red graphs. Set up tables for all three functions. Find the function values for at least 5 values of \( x \). You can use the graphs to estimate the function values.

**Work through Examples 9 and 10 in section 1.4 on page 115-16.**

**Transformations – Shifts (Translations)**

Continuing with the concept of sums of two functions we notice one particularly easy case:
Example: \( f(x) = x^2 \) and \( g(x) = 5 \). The sum \( f(x)+g(x)=x^2+5 \).
Its graph has the same shape as \( f(x) \) but it is moved up by 5 units.
We call this a transformation, because the graph can be viewed as being obtained from the original graph by a geometric operation, i.e. shifting the graph 5 units up.
The effect of adding a constant to or subtracting a constant from $f(x)$ in $y = f(x)$ is a shift of $f(x)$ up or down. Such a shift is called a **vertical translation**.

**Vertical Translation**
For $b > 0$,

- the graph of $y = f(x) + b$ is the graph of $y = f(x)$ shifted up $b$ units;
- the graph of $y = f(x) - b$ is the graph of $y = f(x)$ shifted down $b$ units.
The effect of adding a constant to the $x$-value or subtracting a constant from the $x$-value in $y = f(x)$ is a shift of $f(x)$ to the right or left. Such a shift is called a horizontal translation.

**Horizontal Translation**

For $d > 0$:

- the graph of $y = f(x - d)$ is the graph of $y = f(x)$ shifted right $d$ units;
- the graph of $y = f(x + d)$ is the graph of $y = f(x)$ shifted left $d$ units.

**Symmetry**

The beauty and also the difficulty of this week's topic is that we are combining the algebraic considerations of functions with geometric considerations of their graphs. The idea of a transformation is geometric, as well as the idea of symmetry.

**Discussion Question 3:**

Read the passages about symmetry in section 1.5 starting on page 122. What types of symmetry do we distinguish? How do we decide if the graph of a function exhibits any of these symmetries by looking at the equation of the function? Give different examples from the ones given in the text. What does symmetry have to do with reflections?

**Discussion Question 4:**

Read the passages about odd and even functions in section 1.5 starting on page 125. What does this have to do with symmetry? How does one decide if a function is odd or even or neither? Give different examples from the ones given in the text.

**Transformations – Reflections**
Summarizing the findings from the last discussion questions: The point 
\((x, -y)\) is the reflection of the point \((x, y)\) about the x-axis. 
The point \((-x, y)\) is the reflection of \((x, y)\) about the y-axis. 
The point \((-x, -y)\) is obtained from the point \((x, y)\) by rotating it about the 
origin.

The graph of \(y = -f(-x)\) is obtained from the graph of \(y = f(x)\) by 
rotation about the origin.

Discussion Question 5: 
Work through Example 5 in section 1.5 on page 130. 
Then find the equation of the red graph in the sketch below. That 
function is obtained from the original function by rotating about the 
origin.
Transformations – Stretching and Shrinking

The transformations we have investigated up to now were "rigid", that means they did not change the shapes of the graphs involved. We will now consider stretching and shrinking transformations. One can visualize a stretching of a graph in the direction of the y-axis, by imagining that the graph is drawn on a rubber sheet and then stretched in the direction of the y-axis. Similarly a shrinking corresponds to a squishing of the function in the direction of the y-axis.

Work through the interactive discovery in section 1.5 on page 131.

*Vertical Stretching and Shrinking*

The graph of \( y = af(x) \) can be obtained from the graph of \( y = f(x) \) by

- stretching vertically for \( |a| > 1 \), or
- shrinking vertically for \( 0 < |a| < 1 \).

For \( a < 0 \), the graph is also reflected across the x-axis.
Work through the Examples 6 and 7 in section 1.5 starting on page 132.

You find the summary of transformation on page 134 in the textbook.
**Vertical Translation:** \( y = f(x) \pm b \)

For \( b > 0 \),
- the graph of \( y = f(x) + b \) is the graph of \( y = f(x) \) shifted up \( b \) units;
- the graph of \( y = f(x) - b \) is the graph of \( y = f(x) \) shifted down \( b \) units.

**Horizontal Translation:** \( y = f(x \mp d) \)

For \( d > 0 \),
- the graph of \( y = f(x - d) \) is the graph of \( y = f(x) \) shifted right \( d \) units;
- the graph of \( y = f(x + d) \) is the graph of \( y = f(x) \) shifted left \( d \) units.

**Reflections**

*Across the x-axis:* The graph of \( y = -f(x) \) is the reflection of the graph of \( y = f(x) \) across the x-axis.

*Across the y-axis:* The graph of \( y = f(-x) \) is the reflection of the graph of \( y = f(x) \) across the y-axis.

**Vertical Stretching or Shrinking:** \( y = af(x) \)

The graph of \( y = af(x) \) can be obtained from the graph of \( y = f(x) \) by
- stretching vertically for \( |a| > 1 \), or
- shrinking vertically for \( 0 < |a| < 1 \).

For \( a < 0 \), the graph is also reflected across the x-axis.

**Horizontal Stretching or Shrinking:** \( y = f(cx) \)

The graph of \( y = f(cx) \) can be obtained from the graph of \( y = f(x) \) by
- shrinking horizontally for \( |c| > 1 \), or
- stretching horizontally for \( 0 < |c| < 1 \).

For \( c < 0 \), the graph is also reflected across the y-axis.
Problem Solving

Section 2.1 gives a formal introduction to problem solving. Solving applied problems or "word problems" is unfortunately a nightmare for many students. The good news is that it can be learned, the bad news is that it might take some time and effort, in particular if you have previously avoided them. Many students seem to be able to go through intermediate and beginning algebra classes without a strong exposure to word problems. In some sense, those problems should be really the main topic of those classes, since the algebraic manipulations are useless if one does not understand the context in which they are done. The following is an outline of the steps to solve a challenging problem:

**Five Steps for Problem Solving**

1. **Familiarize** yourself with the problem situation. If the problem is presented in words, then, of course, this means to read carefully. Some or all of the following can also be helpful.
   
   a) Make a drawing, if it makes sense to do so.
   
   b) Make a written list of the known facts and a list of what you wish to find out.
   
   c) Assign variables to represent unknown quantities.
   
   d) Organize the information in a chart or a table.
Work through Example 6 in section 2.1 on page 166.

The third step in solving in this process consists in solving equations. We can also view the process of solving an equation, as the process of finding the zeros of a function, that means finding those x-values, which make the y-value equal to zero. Those are also the x-intercepts of the function involved.

**Zeros of Functions**

An input $c$ of a function $f$ is called a zero of the function, if the output for $c$ is 0. That is, $f(c) = 0$.

Work through Example 1 in section 2.1 on page 160.

Finding zeros of functions is really just a different way of looking at the equation solving process. In this way solving equations, just becomes a
step in the study of the characteristics of functions. For example if \( f(x) = 2x-4 \):

To find the zero of \( f(x) \), we solve \( f(x) = 0 \):

\[
2x - 4 = 0 \\
2x = 4 \\
x = 2.
\]

The solution of \( 2x - 4 = 0 \) is 2. This is the zero of the function \( f(x) = 2x - 4 \). That is, \( f(2) = 0 \).

In future weeks we will study in detail more types of functions and we will learn how one can solve equations involving those functions.