Week 14 –The Binomial Theorem and Review

Many images below are excerpts from the multimedia textbook. You can find them there and in your textbook in section 7.7.

We have nearly come to the end of the material in this course. We have only a tiny bit of new material left and the remainder of this week will be spent with review for the final exam next week.

The Binomial Theorem

We will only cover the first part of section 7.7 until the middle of the page 524.

The Binomial Theorem is a tool to help you expand expressions like \((2x+3)^5\). It turns out that there is a simple pattern, which can be used to find the expansion of any binomial to a power \(n\).

Consider the following expanded powers of \((a + b)^n\), where \(a + b\) is any binomial and \(n\) is a whole number. Look for patterns.

\[
\begin{align*}
(a + b)^0 &= 1 \\
(a + b)^1 &= a + b \\
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{align*}
\]

Let’s just check out the last line. Each term is a product of a number, a coefficient, and a power of \(a\) and a power of \(b\). We notice that the powers of a decrease from 5 to 0 as we are moving from left to right and the powers of \(b\) increase from 0 to 5. All we really need to know is the sequence of the coefficients for a particular row.

Suppose that we want to find an expansion of \((a + b)^6\). The patterns we noted above indicate that there are 7 terms in the expansion:

\[a^6 + c_1a^5b + c_2a^4b^2 + c_3a^3b^3 + c_4a^2b^4 + c_5ab^5 + b^6.\]

We will find the values for the coefficients \(c_1\) through \(c_6\) using Pascal’s Triangle.
(a + b)^0: 1
(a + b)^1: 1 1
(a + b)^2: 1 2 1
(a + b)^3: 1 3 3 1
(a + b)^4: 1 4 6 4 1
(a + b)^5: 1 5 10 10 5 1

There are many patterns in the triangle. Find as many as you can.

Perhaps you discovered a way to write the next row of numbers, given
the numbers in the row above it. There are always 1’s on the outside. Each
remaining number is the sum of the two numbers above it. Let’s try to
find an expansion for (a + b)^6 by adding another row using the patterns
we have discovered:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

We see that in the last row
the 1st and last numbers are 1;
the 2nd number is 1 + 5, or 6;
the 3rd number is 5 + 10, or 15;
the 4th number is 10 + 10, or 20;
the 5th number is 10 + 5, or 15; and
the 6th number is 5 + 1, or 6.

Thus the expansion for (a + b)^6 is

(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.

Discussion Question 1: Find the next two rows in Pascal’s Triangle and use
the triangle to find the expansion of (a + b)^9.
The Binomial Theorem is simply the summary of these facts:

**The Binomial Theorem Using Pascal’s Triangle**

For any binomial \( a + b \) and any natural number \( n \),

\[
(a + b)^n = c_0a^n b^0 + c_1a^{n-1}b^1 + c_2a^{n-2}b^2 + \cdots + c_{n-1}a^1 b^{n-1} + c_n a^0 b^n,
\]

where the numbers \( c_0, c_1, c_2, \ldots, c_{n-1}, c_n \) are from the \((n + 1)\)st row of Pascal’s triangle.

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Read example 1 in section 7.7 on page 523.

Discussion Question 2: Follow the example to find the expansion of \((y – w)^5\).

Discussion Question 3: Find the expansion of \((2 + 3x)^6\).

Read example 2 in section 7.7 on page 524.

Discussion Question 4: Follow the example to find the expansion of \(\left(3w - \frac{2}{w}\right)^4\).

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**Review**

This review will only extend over the material we covered up to the last test. The material from chapter 7 is so new that we will not review it especially. Even though that material will be covered on the final.

The course has 4 major topics
1. Solving equations and inequalities (This includes simplifying algebraic expressions of all types.)
2. Functions and graphs
3. Applications
4. Sequences and Series

We will deal here only with parts 1-3 as explained earlier.

1. Review for solving equations and inequalities.
   a. Solving Equations
The most important step in solving equations is to realize what kind of an equation one is given and what steps are appropriate for each kind of equation.

We have solved: Linear and quadratic equations directly. For Radical and rational equations check out week 5. For exponential and logarithmic equations review weeks 9 and 10. Substitution is an important tool to simplify an equation into a type, which can be solved.

Discussion Question 5: Classify each equation and then solve for x:

a. \( \frac{2}{x^2 - 4} - \frac{1}{x - 2} = \frac{1}{x + 2} \)

b. \( \ln x + \ln (x - 3) = \ln 4 \)

c. \( 3wx - w = 5w + 8x \)

d. \( e^{2x} - 4e^x - 5 = 0 \)

f. \( x^2 + 3x + 1 = x \)

g. \( 8^{2x} = 4 \)

h. \( 3 \ln (x^2 + 1) = 2 \)

i. \( x - \sqrt{2x + 1} = 1 \)

j. \( \frac{1}{x} + \frac{1}{w} = 5 \)

k. \( \left| 2^x - 1 \right| = 2 \)

m. \( x^4 + 6x^2 = 5 \)

n. \( \log_3 x + \log_3 (x - 8) = 2 \)

b. Solving inequalities

The first rule is to convert the inequality into an equality and find the solutions. Now find all values where the function is not defined and add them to the list. These are the values where the inequality can change from true to false or vice versa. The values determine the intervals. You need to find a test value for each interval. The interval is part of the solution set if a value in it makes the inequality true.

This sounds complicated spelled out in detail like this. If you have a grapher available look at the graphs of both sides of the inequality and see where the left hand side graph is above the right hand side graph or vice versa.

Discussion Question 6

Solve each inequality first without and then check with your calculator.

a. \( x^2 - 2x + 2 > 0 \)

b. \( |x + 5| \geq 3 \)

c. \( 10^{x-1} > 0.1 \)

d. \( \log(x - 1) > 2 \)

f. \( \frac{1}{x - 3} \leq 2 \)

g. \( (x+2)^2(x-1) < 0 \)

h. \( e^{3x} < 5 \)

i. \( 5 - (x+3)^2 > 6 \)
2. Functions and graphs
We have studied a variety of different functions and the main properties of their graphs. Learn the facts on the summary sheet by heart. Then go back to week 3 and see how you can obtain other functions by shifts, stretches and reflections of these basic functions. Make up your own examples, write down an equation, sketch it and check on your calculator.
We studied in addition how to graph linear, quadratic, cubic and degree 4 polynomials and rational functions in more detail.

Discussion Question 7
Describe the main properties of the following functions and then sketch the graphs by hand.

a. \( y = -4x + 10 \)  
b. \( y = 2^{-x} - 3 \)  
c. \( y = \frac{2x+1}{x-1} \)  
d. \( y = x^2 - 4x + 3 \)  
f. \( y = \log (-x+3) \)  
g. \( y = \sqrt{5-x} \)

Discussion Question 8
a. Find two functions from different function families each with a vertical asymptote when \( x = 3 \).
b. Find two functions from different function families each with a horizontal asymptote when \( y = 3 \).

Discussion Question 9
Explain how you recognize whether a cubic polynomial has a double zero at \( x = 1 \) or not.

Discussion Question 10
Find two different cubic polynomials with zeros at \( x = 2 \), \( x = -2 \) and \( x = 3 \).

In discussing the properties of functions we also computed inverses of functions provided that they were one-to-one.

Discussion Question 11
Find the inverse function for \( y = \frac{x+1}{x-2} \).

Discussion Question 12
Give an example of a function, which is not one-to-one and explain why.

3. Applications
We studied many applied problems. In particular in week 11, compound interest and radioactive decay. You need to know the formulas for the different cases by heart.
Through out the whole course we saw distance-rate-time and mixture problems with one or two variables. We also solved problems involving percentages and geometrical concepts using the Pythagorean theorem, formulas for the area and circumference of a circle, area of a triangle, area and perimeter of a rectangle and combinations of those concepts.
We expressed certain variables as functions of other variables and used the graphing calculator to find values, which maximize or minimize those functions.

4. Sequences and Series
See weeks 12 and 13 for more details.