Week 12 – Patterns

Many images below are excerpts from the multimedia textbook. You can find them there and in your textbook in section 7.1.

The topic of these last two weeks is quite different from our previous discussions. When we are entering new territory in Mathematics we always have to get used to new notation and the meaning behind the symbols. In some sense that may be the hardest thing about learning Math: to understand what the symbols mean. This week is mainly devoted to learn the symbols and notation of sequences and series and next week we learn how to solve problems with the new concepts.

Sequences

So far we have dealt with functions, which were defined for real numbers. Now we are dealing with functions, which are only defined for natural numbers, the counting numbers 1, 2, 3 etc.

So given a function $f(x) = 2^x$ we were previously interested in the function for its graph and various other properties. Now we are only interested in the function values when $x$ is a positive integer: 2, 4, 8, 16 …

For historical reasons we don’t write this as $f(x)$ any more. As variable we now use the letter $n$, indicating the counting by whole numbers. Instead of $f(x)$ we use $a(n)$ for the function, indicating that we perceive mainly the aspect of the values instead of the function aspect.

So it turns into either $a(n) = 2^n$ or writing the argument as a subscript, we write $a_n = 2^n$. In this vein we will write for $f(1) = 2, f(2) = 4, f(3) = 8$ instead $a_1 = 2, a_2 = 4, a_3 = 8$ etc. This will be the first notational change we have to get used to.

An **infinite sequence** is a function having for its domain the set of positive integers, $\{1, 2, 3, 4, 5, \ldots\}$.

A **finite sequence** is a function having for its domain a set of positive integers, $\{1, 2, 3, 4, 5, \ldots, n\}$, for some positive integer $n$.

We refer to the individual function values as "terms" of the sequence and to the formula as "general term".
Watch the video for section 7.1.

Work through Example 1 in section 7.1 on page 471.
Discussion Question 1: Find the 10th and the 11th term. What does the factor \((-1)^n\) do to the terms? Which terms are positive and which are negative? Why is this type of a sequence called "alternating?"

Finding the general term of a sequence can be quite challenging. In particular if you don't know what type of a function you might be looking for.

Work through Example 3 in section 7.1 on page 472.
Discussion Question 2: Try to explain what clues one could use to come up with the general terms in the parts a and b. What is the general term for the sequence 1, -3, 9, -27, 81,... ? What is the fact illustrated by part c?

Series

Instead of just listing the terms of a sequence we are now interested in adding them up.

**Series**

Given the infinite sequence

\[a_1, a_2, a_3, a_4, \ldots, a_n, \ldots,\]

the sum of the terms

\[a_1 + a_2 + a_3 + \cdots + a_n + \cdots\]

is called an **infinite series**. A **partial sum** is the sum of the first \(n\) terms:

\[a_1 + a_2 + a_3 + \cdots + a_n.\]

A partial sum is also called a **finite series**, or **nth partial sum**, and is denoted \(S_n\).

The concept of adding the terms of a sequence may seem surprising. How can one add up infinitely many terms and obtain something one can work with?
It turns out that this is a very successful thought, which spawned large parts of Calculus. The key idea is that we are adding only parts, which are very small. We will deal with the details in next week. In fact we will learn to compute the value of such infinite sums. This week we confined ourselves to just getting used to the notation, since that alone is a challenge.

Work through Example 5 in section 7.1 on page 473.
Discussion Question 3: Find \( S_3 \) and \( S_5 \) for the sequence 3, 5, 7, 9, ..

The convenient way to express sums with many elements is to use the sigma notation. The Greek letter sigma stands for "S" denoting "Sum". We can read it like a computer instruction: \( \sum_{k=1}^{5} (2k+1) \) means "For \( k \) starting at 1, increasing by 1 in each step, compute the values \( (2k+1) \) and then add them up until \( k = 5 \)."

The sum does not always start out when \( k = 1 \), it could start out when \( k = 0 \). \( \sum_{k=0}^{4} (2k+3) \) is the summation of the terms where \( k \) starts at \( k = 0 \) and ranges until \( k = 4 \).

Discussion Question 4: List explicitly the elements in the sums \( \sum_{k=1}^{5} (2k+1) \) and \( \sum_{k=0}^{4} (2k+3) \). Find a way to express the first sum as starting when \( k = 2 \).

When you have to express a given sum in sigma notation you have to first find the general term of the sequence of the elements of the sum.

Work through Example 8 in section 7.1 on page 475.
Discussion Question 5: Find a solution for part a.) where the sum starts at \( k =1 \). Find a solution for part b.) where the sum starts at \( k = 0 \) and find a solution for part c.) where the sum starts out at \( k = 2 \).

Sequences defined recursively

In recursively defined sequences you are given a first term and then a rule to compute the next term from the previous term. This is sometimes
easier to see than the general term. We will need to learn how to find the general term in these situations.

Work through Example 9 in section 7.1 starting on page 475.
Discussion Question 6: Find the first 5 terms of the sequence when \(a_1=4\) and \(a_{k+1}=2a_k-1\).