Week 11 –Applications

Many images below are excerpts from the multimedia textbook. You can find them there and in your textbook in sections 4.6 and 5.2. Exponential functions are used to model a certain type of growth function called "exponential growth". It is characterized by the fact that the percentage growth per year, or other fixed time period is always a fixed percentage.

Compound Interest

The computation of interest is a good way to investigate the different ways growth can occur. When you invest $1000 into a bank account you are paid a certain amount of interest. Say the bank advertises that they pay 6% yearly interest. Assume that you do not withdraw any amount from the account. We will investigate what different possibilities exist for your money to grow after you have deposited the $1000 on January 1, 2000; that means we will find a function \( P(t) \) describing the amount of money in your account after \( t \) years after the year 2000.

1. **Simple interest.** In this situation, the bank only pays interest on the original $1000. So the amount of interest you receive in each year is only $60 no matter how many years you wait. \( P(t) = 1000 + 60t \). This is a linear function. It usually does NOT appear in a financial context, unless very special arrangements have been made, say in a loan from a parent to a child or a similar "friendly" loan situation.

2. **Yearly compounded interest.** In this situation the bank pays 6% on the amount in the account present during a particular year. On December 31, 2000 you are paid $60, and so your balance during the year 2001 is $1060, and in December 2001, you are paid $63.60. You can think that this is composed of $60 of interest on the original $1000 and $3.60 of interest on the interest you earned in your first year. So during the year 2002, you have $1,123.60 in your account and in December 2002 your interest payment is $67.42.
<table>
<thead>
<tr>
<th>Year t years after 2000</th>
<th>2000 t=0</th>
<th>2001 t=1</th>
<th>2002 t=2</th>
<th>2003 t=3</th>
<th>In general</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount in account on Jan. 1 of the year</td>
<td>1000</td>
<td>1060</td>
<td>1,123.60</td>
<td>1,191.02</td>
<td>1000(1+0.06)^t</td>
</tr>
</tbody>
</table>

One way to come up with a formula for this pattern is to realize that in each case you take the old balance and add 6% to it, or in symbols: 
Balance + 0.06 Balance = Balance (1 + 0.6). So you get a factor (1+0.06) for each year you leave the money in the account. That means the number of years, t, becomes the exponent of the expression.

When interest is compounded yearly at a yearly rate of r the balance after t years of an initial investment of P₀ Dollars is given by the formula: 

\[ P(t) = P₀(1+r)^t. \]

Discussion Question 1: You invest $5000 at an interest rate 7.5% compounded annually. a. What is the balance after 10 years? b. How much interest was earned during the 10 years? c. After how many years is the balance $20,000?

Yearly compounded interest is usually not found in the real world either. Many years ago, savings booklets may have carried this type of interest. Nowadays, interest is computed more often.

3. Monthly compounded interest. Let us go back to our example. You invest $1000 on January 1, 2000, and get 6% annual interest. But now you receive your first interest payment at the end of January. Now since you did not leave the money in the account for a whole year but only for one month, or 1/12 of a year you should not receive all of the $60 of interest either, but only 1/12 of the interest, which is $5. But now during the month of February you have a balance of $1,005, so your interest based on 6% for one month would by 1/12 of $60.30, which is $5.025. Which means that during the month of March your interest is based on a balance of $1,010.25 and your interest credited at the end of March is $5.05. You see that you will get more interest during the whole year, than the $60, since the $5.00 you received in January will earn interest for 11 months, and the $5.025 earned in February will earn interest for
10 months and so on. The formula describing the total balance at the end of t years is obtained similarly to the previous situation. You now basically have a "compounding period" of one month instead of one year, and the interest rate for this compounding period is 1/12 of the 6%. So
\[ P(t) = 1000 \left(1 + 0.06/12\right)^{12t}. \]

In many situations the compounding period is even shorter than a month. Credit card companies use a daily compounding for example. We will use the letter n to describe the number of times interest is compounded during a particular year. So in the case of monthly compounding n = 12 and for daily compounding n = 365.

When interest is compounded n times a year at a yearly nominal rate of r the balance after t years of an initial investment of \( P_0 \) Dollars is given by the formula:
\[ P(t) = P_0(1+r/n)^{nt}. \]

Work through Example 9 in section R2 starting on page 25. If $2,500 are invested at 5.5% compounded quarterly, what is the balance after 10 years?

Why did we use the term "yearly nominal rate"? In our example of monthly compounding the balance after one year, or 12 months, is $1000 \( (1+0.06/12)^{12} \) = $1,061.68. But that means that the actual yearly interest was not 6% but rather 6.168%. This interest rate is called the "effective interest rate". It is the actual rate, at which the balance grows per year. The 6% has to be distinguished from this and is therefore called the nominal rate. Banks have to disclose the effective rate when they make loans and it is sometimes called APY (annual percentage yield). The easiest way to compute the effective interest rate is to find the balance after 1 year, when $100 is invested. The difference between the original $100 and the balance is exactly the effective interest.

Discussion Question 2: You invest $5000 at an interest rate 7.5% compounded monthly. a. What is the balance after 10 years? b. How much interest was earned during the 10 years? c. After how many years is the balance $20,000? d. What is the effective interest rate for this investment?
Work through Example 5 in section 4.2 starting on page 303. Use the method from the example to compute the amount of time needed for an investment of $1000 at a nominal rate of 8% compounded semi-annually to grow to $50,000.

4. Interest compounded continuously. The more often interest is compounded in a year the higher is the effective interest rate and the balance.

Discussion Question 3: Compute the balance after one year when $1000 is invested at 8%

a. Compounded yearly  b. Compounded monthly  c. Compounded daily
d. Compounded hourly  e. Compounded every second.

The balance gets higher every time we increase the number of times interest is compounded per year, but the balance does not increase without bounds. That means there is a number, which cannot be surpassed, no matter how often the interest is compounded per year. That number is $1000 \cdot e^{0.08}$. This is in fact the reason why the number $e$ is so important. So we get a last way to compound interest:

When interest is compounded continuously a yearly nominal rate of $r$ the balance after $t$ years of an initial investment of $P_0$ Dollars is given by the formula:

$$P(t) = P_0 \cdot e^{rt}.$$

Work through Example 2 in section 4.6 starting on page 343. Use the techniques from the example to answer questions a)-d) when $2000 grow to $3,500 in 5 years.

What effective interest rate corresponds to 9.3% compounded continuously?

In some sense there is really just one exponential function, even though we have just developed all these different formulas. We can express $P(t) = 1000 \cdot (1+0.06)^t$ as $P(t)=1000 \cdot e^{t \ln(1.06)}$ and in the same way could we express monthly compounding in terms of the
exponential function $e^{kt}$. For that reason we usually only speak of this function, when we are dealing with exponential growth. In reality they are all the same.

**Exponential Growth and Decay**

The function $P(t) = P_0 e^{kt}$ is called exponential growth when $k > 0$ and exponential decay when $k < 0$. In the case of decay we often write it as $P(t) = P_0 e^{-kt}$ for $k > 0$.

As we observed above, exponential growth is really the same as interest compounded continuously. You have to realize that $k$ is not the actual (effective) rate of growth, it is the "nominal" rate of growth.

One of the characteristic features of exponential growth and decay is that the time to grow or decay a fixed percentage, any fixed percentage, is always constant, no matter how large or small the function value is.

Of particular interest is the time to grow by 100%, or to double, the doubling time, and the time to decay 50%, or the half-life.

**Watch the video for section 4.6.**

Work through Example 1 in section 4.6 starting on page 341. Use the techniques from the example to answer questions a)-d) if the growth rate is 1.7%.

Work through Example 3 in section 4.6 on page 345. What is the effective rate of growth per year?

Work through Example 5 in section 4.6 starting on page 347. How old is a sample, which has lost 30% of its carbon-14 at the time it was found?

Since exponential growth has no bounds and speeds up tremendously as time goes by, it has become synonymously with rapid growth. You will hear the phrase "grows exponentially" used in the media. Very often exponential growth is not a good model for growth situations where a limit has to be reached eventually. In this case we might use a logistic function.
Solving Linear Systems with Three Variables

We have previously encountered systems of two linear equations in two variables. We used the elimination method or substitution.

When we have to solve a system of three linear equations with three variables, it is often helpful to have a systematic approach.

Watch the video for section 5.2 or alternatively check out example 1 in section 5.2 starting on page 371. Use the method to solve exercise 3 on page 377.